

Quiz 2: MATH 212

Instructors: Mohammad El Smaily & Wael Mahboub November 6, 2015

Duration: 70 minutes

Name (Last, First):		
Student number:		
Circle your instructor's name and your section's number:		
M. El Smaily	Section: 1 (from 11:00 am to 12:00 pm)	

Section 3 (from 1:00pm to 2:00pm)

W. Mahboub Section: 2 (from 12:00 pm to 1:00 pm),

For marker's use only		
Problem	Score	
1	/10	
2	/20	
3	/15	
4	/40	
5	/15	
Total	/100	
	,	

[10 points] Problem 1. Consider the function F(x) defined over \mathbb{R} by

$$F(x) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{7}\right) + b_n \sin\left(\frac{n\pi x}{7}\right),$$

where, for each $n \geq 0$, a_n and b_n are given by

$$a_n = \frac{1}{7} \int_{-7}^7 e^{-4x} \cos\left(\frac{n\pi x}{7}\right) dx$$
 and $b_n = \frac{1}{7} \int_{-7}^7 e^{-4x} \sin\left(\frac{n\pi x}{7}\right) dx$.

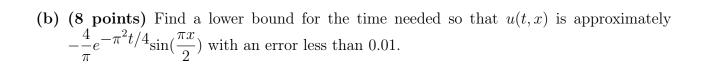
(a) (5 points) Find F(0). Then, deduce the value of the sum $\sum_{n=0}^{\infty} a_n$. (Justify your answers).

(b) (5 points) Find F(7) and then deduce the value of the sum $\sum_{n=0}^{\infty} (-1)^n a_n$.

 $[20 \ points=12+8]$ Problem 2. Consider the heat problem with homogeneous Dirichlet boundary conditions

$$u_t = u_{xx}$$
 for $t > 0$, $0 < x < 2$; $u(t, 0) = u(t, 2) = 0$; $u(0, x) = x$ for $0 \le x \le 2$.

(a) (12 points) Write down the solution u(t, x) in a series format.



[15 points] Problem 3. Consider the sequence of continuous functions

$$f_n(x) = \begin{cases} 1, & x \ge 0 \\ nx + 1, & -\frac{1}{n} \le x < 0 \\ 0, & x < -\frac{1}{n}. \end{cases}$$

(a) (8 points) Find the pointwise limit f(x) over \mathbb{R} and sketch its graph.

(b) (7 points) Does $\{f_n(x)\}_n$ converge uniformly to f over \mathbb{R} ? Justify your answer.

[40 points] Problem 4. Let u(t,x) be the solution to the heat problem

$$u_t(t, x) = u_{xx} + bu + c, \quad t > 0, \quad 0 < x < 1$$

 $u(t, 0) = u(t, 1) = 0, \quad t > 0$
 $u(0, x) = 400,$

where b and c are two constants. Denote the equilibrium solution of the above problem by $u_E(x)$ and let

$$v(t,x) = e^{-bt}(u(t,x) - u_E(x)).$$

(a) (10 points) Show that v(t,x) satisfies the simpler PDE

$$v_t = v_{xx}$$
, for all $t > 0$, $x \in (0,1)$ with boundary conditions $v(t,0) = v(t,1) = 0$.

(b) (10 points) Set $b = \frac{\pi^2}{4}$. Show that equilibrium solution $u_E(x)$ is given by

$$u_E(x) = \frac{4c}{\pi^2} \left[\cos\left(\frac{\pi x}{2}\right) + \sin\left(\frac{\pi x}{2}\right) - 1 \right]$$

Help: you will need to solve an **inhomogeneous** ordinary differential equation (after setting $b = \frac{\pi^2}{4}$).

(c) (15 points) You can assume the result of part (a) here. Write down v(t,x) in a series format.

Help: make sure that you compute the initial condition v(0,x) in order to find the coefficients of the series. You may also use the formulæ

$$\cos\left(\frac{\pi x}{2}\right)\sin n\pi x = \frac{1}{2}\left[\sin(n+1/2)\pi x + \sin(n-1/2)\pi x\right] \text{ and }$$

$$\sin\left(\frac{\pi x}{2}\right)\sin n\pi x = \frac{1}{2}\left[\cos(n-1/2)\pi x - \cos(n+1/2)\pi x\right].$$

(d) (5 points) Conclude a series formula for the solution u(t,x) of the original problem, where $b = \frac{\pi^2}{4}$.

[15 points] Problem 5. Compute the Fourier series of the function sign(x) defined by 1 for $x \ge 0$, and -1 for x < 0.