# Quiz 2: MATH 212

Instructors: Mohammad El Smaily & Wael Mahboub

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**Duration: 70 minutes**

**Name (Last, First):** ______________________________

**Student number:** ______________________________

**Circle your instructor's name and your section's number:**

**M. El Smaily** Section: 1 (from 11:00 am to 12:00 pm)

**W. Mahboub** Section: 2 (from 12:00 pm to 1:00pm ),

Section 3 (from 1:00pm to 2:00pm)

| For marker's use only | |
|---|---|
| Problem | Score |
| 1 | /10 |
| 2 | /20 |
| 3 | /15 |
| 4 | /40 |
| 5 | /15 |
| Total | /100 |

**[10 points] Problem 1.** Consider the function $F(x)$ defined over $\mathbb{R}$ by

$$F(x) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{7}\right) + b_n \sin\left(\frac{n\pi x}{7}\right),$$

where, for each $n \geq 0$, $a_n$ and $b_n$ are given by

$$a_n = \frac{1}{7}\int_{-7}^{7} e^{-4x} \cos\left(\frac{n\pi x}{7}\right) dx \quad \text{and} \quad b_n = \frac{1}{7}\int_{-7}^{7} e^{-4x} \sin\left(\frac{n\pi x}{7}\right) dx.$$

**(a) (5 points)** Find $F(0)$. Then, deduce the value of the sum $\sum_{n=0}^{\infty} a_n$. (Justify your answers).

**(b) (5 points)** Find $F(7)$ and then deduce the value of the sum $\sum_{n=0}^{\infty} (-1)^n a_n$.

**[20 points=12+8] Problem 2.** Consider the heat problem with homogeneous Dirichlet boundary conditions

$$u_t = u_{xx} \text{ for } t > 0,\ 0 < x < 2; \quad u(t,0) = u(t,2) = 0; \quad u(0,x) = x \text{ for } 0 \le x \le 2.$$

**(a) (12 points)** Write down the solution $u(t,x)$ in a series format.

**(b) (8 points)** Find a lower bound for the time needed so that $u(t,x)$ is approximately $-\frac{4}{\pi}e^{-\pi^2 t/4}\sin(\frac{\pi x}{2})$ with an error less than 0.01.

**[15 points] Problem 3.** Consider the sequence of continuous functions

$$f_n(x) = \begin{cases} 1, & x \geq 0 \\ nx + 1, & -\dfrac{1}{n} \leq x < 0 \\ 0, & x < -\dfrac{1}{n}. \end{cases}$$

**(a) (8 points)** Find the pointwise limit $f(x)$ over $\mathbb{R}$ and sketch its graph.

**(b) (7 points)** Does $\{f_n(x)\}_n$ converge uniformly to $f$ over $\mathbb{R}$? Justify your answer.

**[40 points] Problem 4.** Let $u(t,x)$ be the solution to the heat problem

$$u_t(t,x) = u_{xx} + bu + c, \quad t > 0, \quad 0 < x < 1$$
$$u(t,0) = u(t,1) = 0, \quad t > 0$$
$$u(0,x) = 400,$$

where $b$ and $c$ are two constants. Denote the equilibrium solution of the above problem by $u_E(x)$ and let

$$v(t,x) = e^{-bt}(u(t,x) - u_E(x)).$$

**(a) (10 points)** Show that $v(t,x)$ satisfies the simpler PDE

$$v_t = v_{xx}, \text{ for all } t > 0, \ x \in (0,1) \text{ with boundary conditions } v(t,0) = v(t,1) = 0.$$

**(b)** **(10 points)** Set $b = \dfrac{\pi^2}{4}$. Show that equilibrium solution $u_E(x)$ is given by

$$u_E(x) = \frac{4c}{\pi^2}\left[\cos\left(\frac{\pi x}{2}\right) + \sin\left(\frac{\pi x}{2}\right) - 1\right]$$

**Help:** you will need to solve an **inhomogeneous** ordinary differential equation (after setting $b = \dfrac{\pi^2}{4}$).

**(c) (15 points)** You can assume the result of part (a) here. Write down $v(t,x)$ in a series format.

**Help:** make sure that you compute the initial condition $v(0,x)$ in order to find the coefficients of the series. You may also use the formulæ

$$\cos\left(\frac{\pi x}{2}\right)\sin n\pi x = \frac{1}{2}\left[\sin(n+1/2)\pi x + \sin(n-1/2)\pi x\right] \text{ and}$$

$$\sin\left(\frac{\pi x}{2}\right)\sin n\pi x = \frac{1}{2}\left[\cos(n-1/2)\pi x - \cos(n+1/2)\pi x\right].$$

**(d) (5 points)** Conclude a series formula for the solution $u(t, x)$ of the original problem, where $b = \dfrac{\pi^2}{4}$.

**[15 points] Problem 5.** Compute the Fourier series of the function $\text{sign}(x)$ defined by 1 for $x \geq 0$, and $-1$ for $x < 0$.